# L11: Hoare Logic

**Software Analysis**

IIT Guwahati · August 2026

# Example

A program is supposed to satisfy some property. Consider one statement:

```
x := x + 1;
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Hoare, *An Axiomatic Basis for Computer Programming*, CACM 12(10), 1969

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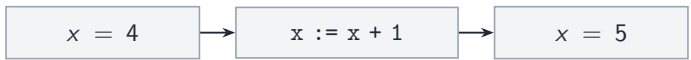


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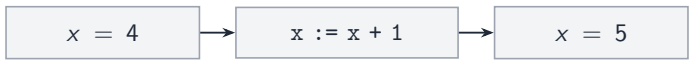


How do we **state** this, and **prove** it, without running the program?

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# Program State

A **state** is the current value of every program variable: $\sigma : \mathit{Var} \to \mathbb{Z}$

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# Program State

A **state** is the current value of every program variable: $\sigma : \mathit{Var} \to \mathbb{Z}$

$$\sigma = \{\, x \mapsto 4,\ y \mapsto 10 \,\}$$

A command transforms one state into another:


Only x changes; every other variable keeps its value.

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# Assertions

An **assertion** is a logical statement about a state. Examples:

$$x > 0 \qquad x = y + 1 \qquad x \geq 0 \wedge y \geq 0 \qquad x \neq y$$

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$$\sigma \models x < y \qquad\qquad \sigma \not\models x > y$$

$P \Rightarrow Q$ means: in *every* state, if $P$ holds then $Q$ holds.

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- $P$ is what we **assume** about the initial state
- $Q$ is what we **guarantee** about the final state

For `x := x + 1`: $P \equiv x = 4$ and $Q \equiv x = 5$.

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# The Hoare Triple

$$\{ P \}\ C\ \{ Q \}$$

If $P$ holds before executing $C$, and $C$ terminates, then $Q$ holds afterwards.

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# Partial Correctness

The triple $\{P\}\ C\ \{Q\}$ holds if and only if:

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- and that execution terminates,
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- $\{x = 4\}\ x := x + 1\ \{x = 5\}$ **holds**
- $\{x = 4\}\ x := x + 1\ \{x = 6\}$ **does not hold**
- $\{X = 1\}$ `while` $X > 0$ `do` $X := X + 1\ \{X = 0\}$ **holds** — the loop never terminates

Partial correctness says *nothing* about termination.

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# Total Correctness

$$[P]\ C\ [Q]$$

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# Total Correctness

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- whenever $C$ is executed in an initial state satisfying $P$,
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$$\text{total correctness} = \text{partial correctness} + \text{termination}$$

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# How Do We Prove a Triple?

A program is built from a few constructs, so we give one **inference rule** per construct:

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$$\frac{A \qquad B}{C}$$

if $A$ and $B$ hold, then we may conclude $C$

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# Inference Rules for Hoare Triples

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If $C_1$ takes us from $P$ to $R$, and $C_2$ takes us from $R$ to $Q$, then $C_1; C_2$ takes us from $P$ to $Q$.

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# Rule 1: `skip`

`skip` does nothing: it leaves the state unchanged.

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Whatever was true before is still true afterwards — for *every* $P$.

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# Rule 2: Assignment

`x := E` evaluates $E$ in the current state and stores the result in `x`. Nothing else changes.

```
x = 100;   y = 4;
x := y + 1;          // now x = 5, y = 4
```

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$$\{\,?\,\}\; x := x + 2\; \{\,x > 10\,\}$$

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$$\{\,?\,\}\; x := x + 2\; \{\,x > 10\,\}$$

What must hold *before* the assignment, for $x > 10$ to hold *after* it?

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# Assignment: Think Backwards

- We want $x > 10$ to hold *after* `x := x + 2`

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- So we need $x + 2 > 10$ to hold *before*, that is $x > 8$

$$\{\, x > 8 \,\}\ x := x + 2\ \{\, x > 10 \,\}$$

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$$\{\, x > 8 \,\}\ x := x + 2\ \{\, x > 10 \,\}$$

The assignment rule is **backward reasoning**: the precondition is computed from the postcondition.

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# The Assignment Rule

$$\frac{}{\vdash \{\, Q[E/x] \,\}\ x := E\ \{\, Q \,\}}$$

- $Q[E/x]$ means: replace every free occurrence of $x$ in $Q$ by $E$

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- With $Q \equiv x > 10$ and $E \equiv x + 2$: $Q[E/x] \equiv x + 2 > 10 \equiv x > 8$

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- With $Q \equiv x > 10$ and $E \equiv x + 2$: $Q[E/x] \equiv x + 2 > 10 \equiv x > 8$
- Hence $\{\, x + 2 > 10 \,\}\ x := x + 2\ \{\, x > 10 \,\}$

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# Rule 3: Sequence

```
x := x + 1;
y := x + 1;
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- $\{ x = 4 \}\ x := x + 1\ \{ x = 5 \}$

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- $\{ x = 4 \}\ x := x + 1\ \{ x = 5 \}$
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- therefore $\{ x = 4 \}\ x := x + 1;\ y := x + 1\ \{ y = 6 \}$

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$$\frac{\vdash \{P\}\ C_1\ \{R\} \qquad \vdash \{R\}\ C_2\ \{Q\}}{\vdash \{P\}\ C_1; C_2\ \{Q\}}$$

$R$ is the **intermediate assertion**: the postcondition of $C_1$ and the precondition of $C_2$.

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# Rule 4: Conditional

```
if (x > 0) y := 1; else y := 2;
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```
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$$\frac{\vdash \{P \wedge B\}\ C_1\ \{Q\} \qquad \vdash \{P \wedge \neg B\}\ C_2\ \{Q\}}{\vdash \{P\}\ \texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ \{Q\}}$$

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# Rule 5: Consequence

$$\frac{P \Rightarrow P' \qquad \vdash \{P'\}\ C\ \{Q'\} \qquad Q' \Rightarrow Q}{\vdash \{P\}\ C\ \{Q\}}$$

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- A precondition may always be **strengthened**
- A postcondition may always be **weakened**

Example: from $\{x > 8\}\, x := x + 2\, \{x > 10\}$ we obtain $\{x = 20\}\, x := x + 2\, \{x > 0\}$.

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# Loops

```
while (x > 0) x := x - 1;
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- Unlike `if`, the body may execute $0, 1, 2, 3, \ldots$ times

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- Unlike `if`, the body may execute $0, 1, 2, 3, \ldots$ times
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- We need one property that holds however many iterations have run

**The loop invariant**

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# Loop Invariant Example

```
// pre:  x >= 0
while (x > 0) x := x - 1;
// post:  x == 0
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$$I \equiv x \geq 0$$

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Note $x > 0$ is **not** an invariant: it is false on the last visit to the loop test.

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# The While Rule

$$\frac{\vdash \{I \wedge B\}\ C\ \{I\}}{\vdash \{I\}\ \texttt{while}\ B\ \texttt{do}\ C\ \{I \wedge \neg B\}}$$

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# Definition: Loop Invariant

**Definition.** $I$ is a **loop invariant** of `while` $B$ `do` $C$ if and only if

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**Loop head** = the program point immediately *before* each evaluation of the guard $B$.

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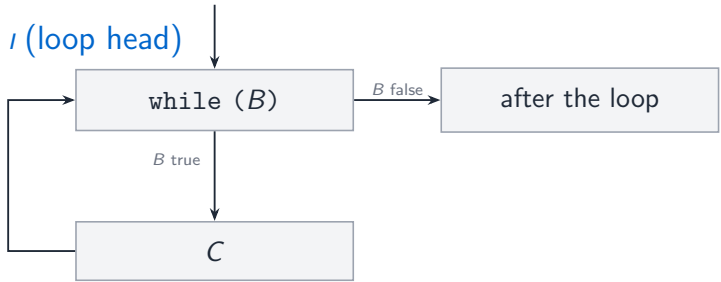
# Three Points, Three Assertions

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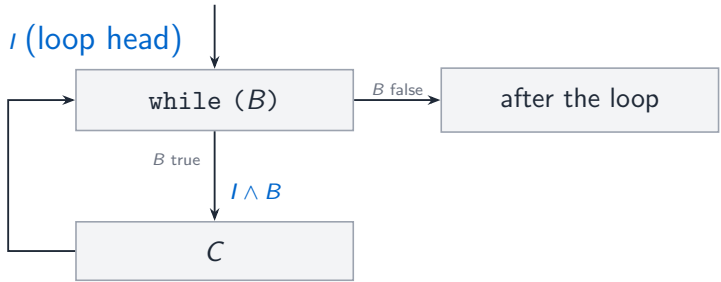


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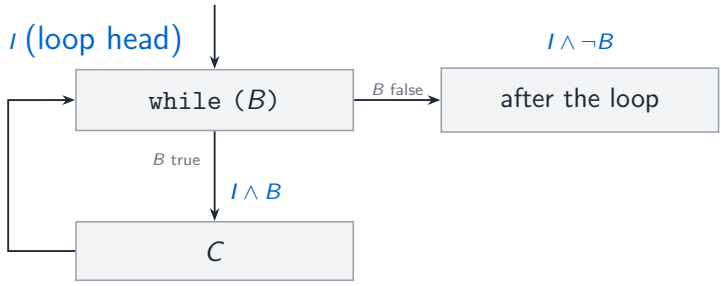


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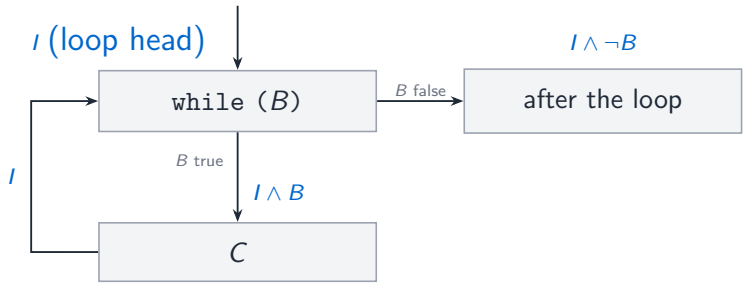


$I$ may be *false* in the middle of $C$; it must be true again by the end of $C$.

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# Using an Invariant: Three Obligations

$$\frac{P \Rightarrow I \qquad \vdash \{I \wedge B\}\ C\ \{I\} \qquad I \wedge \neg B \Rightarrow Q}{\vdash \{P\}\ \texttt{while}\ B\ \texttt{do}\ C\ \{Q\}}$$

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1. **Initialisation**: $I$ holds when the loop head is first reached

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1. **Initialisation**: $I$ holds when the loop head is first reached
2. **Preservation**: the body re-establishes $I$

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# Using an Invariant: Three Obligations

$$\frac{P \Rightarrow I \qquad \vdash \{I \wedge B\}\ C\ \{I\} \qquad I \wedge \neg B \Rightarrow Q}{\vdash \{P\}\ \texttt{while}\ B\ \texttt{do}\ C\ \{Q\}}$$

1. **Initialisation**: $I$ holds when the loop head is first reached
2. **Preservation**: the body re-establishes $I$
3. **Exit**: $I$ together with $\neg B$ implies the postcondition

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# `for` Loop Example

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for (int i = 0; i < 10; i++) { /* body */ }
```

is by definition the same program as

```
int i = 0;
while (i < 10) {            // loop head:  the invariant is asserted here
    /* body */
    i = i + 1;              // the update belongs to the loop body
}
// after the loop
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Guard $B \equiv i < 10$; body $C \equiv \text{body}; i := i + 1$.

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# What Is True at the Loop Head

| visit to the head | value of `i` | guard $i < 10$ | action |
|---|---|---|---|
| 1 | 0 | true | run body, then `i++` |
| 2 | 1 | true | run body, then `i++` |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | 9 | true | run body, then `i++` |
| 11 | **10** | **false** | exit the loop |

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- At the head, `i` takes every value in $\{0, 1, \ldots, 10\}$ — **eleven** visits, not ten
- So any invariant must be true when $i = 10$

# The Loop Invariant

$$I \equiv 0 \leq i \leq 10$$

- **Initialisation**: after $i := 0$ we have $0 \leq 0 \leq 10$ **holds**

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- **Initialisation**: after $i := 0$ we have $0 \leq 0 \leq 10$ **holds**
- **Preservation**: $I[i+1/i] \equiv 0 \leq i+1 \leq 10 \equiv -1 \leq i \leq 9$.
  Since $I \wedge B \equiv (0 \leq i \leq 10) \wedge (i < 10)$, we have $I \wedge B \Rightarrow I[i+1/i]$. **Hence preservation holds.**

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  Since $I \wedge B \equiv (0 \leq i \leq 10) \wedge (i < 10)$, we have $I \wedge B \Rightarrow I[\,i+1/i\,]$. **Hence preservation holds.**
- **Exit**: $I \wedge \neg B \equiv (0 \leq i \leq 10) \wedge (i \geq 10) \equiv i = 10$ **holds**

The exit obligation yields exactly the fact we want after the loop: $i = 10$.

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